1 UNIFIED ANALYTICAL SOLUTION FOR RADIAL FLOW TO A WELL IN A

2 CONFINED AQUIFER

Phoolendra Kumar Mishra and Velimir V. Vesselinov

- 5 Earth and Environmental Sciences Division
- 6 Los Alamos National Laboratory
- 7 MS T003, Los Alamos, NM 87544 USA

3

9 ABSTRACT

Drawdowns generated by extracting water from a large diameter (e.g. water supply) well are affected by wellbore storage. We present an analytical solution in Laplace transformed space for drawdown in a uniform anisotropic aquifer caused by withdrawing water at a constant rate from a partially penetrating well with storage. The solution is back transformed into the time domain numerically. When the pumping well is fully penetrating our solution reduces to that of *Papadopulos and Cooper* [1967]; *Hantush* [1964] when the pumping well has no wellbore storage; *Theis* [1935] when both conditions are fulfilled and *Yang et.al.* [2006] when the pumping well is partially penetrating, has finite radius but lacks storage. We use our solution to explore graphically the effects of partial penetration, wellbore storage and anisotropy on time evolutions of drawdown in the pumping well and in observation wells.

22

23

24

25

26

27

28

29

30

31

32

33

34

35

36

37

38

39

40

41

42

When water is pumped from a large diameter (e.g. water supply) well drawdown in the surrounding aquifer is affected by temporal decline in wellbore storage. An analytical solution accounting for this effect under radial flow toward a fully penetrating well of finite diameter with storage was developed by Papadopulos and Cooper [1967]. A corresponding solution without wellbore storage was presented earlier by van Everdingen and Hurst [1949] and later. in elliptical coordinates, by Kucuk and Brigham [1979]. Mathias and Butler [2007] extended the solution of Kucuk and Brigham [1979] by adding wellbore storage and horizontal anisotropy. Their solution utilized Mathieu functions in Laplace transformed space and numerical inversion of the result into the time domain. Yang et.al. [2006] extended the solution of van Everdingen and Hurst [1949] by allowing the pumping well to be partially penetrating. Dougherty and Babu [1984] developed an analytical solution for a pumping well with storage in a confined double porosity reservoir. Their solution can be reduced to that for a single porosity confined aquifer but ignores anisotropy. None of the available analytical solutions account simultaneously for aquifer anisotropy, partial penetration and storage capacity of the pumping well under confined aquifer conditions.

Moench [1997, 1998] developed an analytical solution for flow to a pumping well with storage in a uniform anisotropic unconfined (water table) aquifer. We present a new solution for radial flow to a partially penetrating well of finite diameter with storage in an anisotropic confined aquifer. Whereas Moench [1997, 1998] used Fourier cosine series in Laplace transformed space we employ Laplace transformation with respect to time followed by finite cosine transformation with respect to vertical coordinates. Our solution reduces to that of Papadopulos and Cooper [1967] when the pumping well is fully penetrating, Hantush [1964] in

the absence of wellbore storage, *Theis* [1935] when both conditions are fulfilled, and *Yang et.al*. [2006] when the pumping well is partially penetrating, has finite radius but lacks storage. We use our solution to explore graphically the effects of partial penetration, wellbore storage and anisotropy on time evolutions of drawdown in the pumping well and in observation wells.

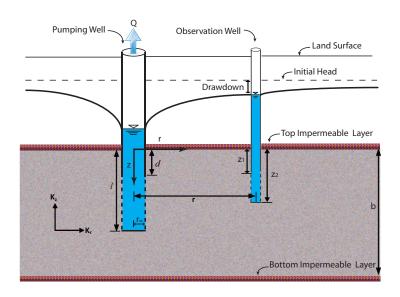


Figure 1: Schematic representation of system geometry

49 THEORY

Problem Definition

Consider a well of finite radius r_w that is in hydraulic contact with a surrounding confined aquifer at depths d through l below the top (Figure 1). The aquifer is horizontal and of infinite lateral extent with uniform thickness b, uniform hydraulic properties and anisotropy ratio $K_D = K_z/K_r$ between vertical and horizontal hydraulic conductivities, K_z and K_r , respectively. Initially, drawdown s(r,z,t) throughout the aquifer is zero where r is radial distance from the axis of the well, z is depth below the top of the aquifer and t is time. Starting

- at time t = 0 water is withdrawn from the pumping well at a constant volumetric rate Q.
- Consider the bottom of the well to be impermeable and ignore flow beneath it. Then drawdown
- 59 distribution in space-time is controlled by

$$60 K_r \left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} \right) + K_z \frac{\partial^2 s}{\partial z^2} = S_s \frac{\partial s}{\partial t} 0 \le z < b (1)$$

61 subject to

62
$$s(r,z,0) = s(\infty,z,t) = 0$$
 $r \ge r_w$ (2)

63
$$\frac{\partial s}{\partial z} = 0$$
 at $z = 0$ & $z = b$ $r > r_w$

64 (3)

66
$$r\left(\frac{\partial s}{\partial r}\right) = 0$$
 $0 < z < d \& l < z < b$ (5)

- where C_w is wellbore storage coefficient (volume of water released from well storage per unit
- drawdown in it).

69 Solution in Laplace Space

- We show in Appendix A that the Laplace transform of the solution, indicated by an
- 71 overbar, is given by

$$\overline{s}(r_{D}, z_{D}, p_{D}) = \frac{Qt}{4\pi K_{r}b} \left\{ \frac{2}{p_{D}} \frac{K_{0}(\phi_{0})}{r_{wD}\phi_{0}K_{1}(r_{wD}\phi_{0}) + \frac{C_{wD}}{2(l_{D} - d_{D})} r_{wD}^{2}\phi_{0}^{2}K_{0}(r_{wD}\phi_{0})} \right\}$$

$$4 \sum_{n=0}^{\infty} K_{0}(\phi_{n}) \left[\sin(n\pi l_{D}) - \sin(n\pi d_{D}) \right] \cos(n\pi z_{D}) / n$$
(6)

$$+\frac{4}{p_{D}\pi(l_{D}-d_{D})}\sum_{n=1}^{\infty}\frac{K_{0}(\phi_{n})[\sin(n\pi l_{D})-\sin(n\pi d_{D})]\cos(n\pi z_{D})/n}{r_{wD}\phi_{n}K_{1}(r_{wD}\phi_{n})+\frac{C_{wD}}{2(l_{D}-d_{D})}r_{wD}^{2}\phi_{0}^{2}K_{0}(r_{wD}\phi_{n})}\right\}$$

where $r_D = r/b$, $z_D = z/b$, $p_D = pt$, $r_{wD} = r_w/r$, $C_{wD} = C_w/(\pi S_s r_w^2)$, $d_D = d/b$, $l_D = l/b$,

74 $\phi_n = \sqrt{p_D/t_s + \beta^2 n^2 \pi^2}$, $t_s = \alpha_s t/r^2$, $\alpha_s = K_r/S_s$ and $\beta = r_D K_D^{-1/2}$, K_0 and K_1 being modified

75 Bessel functions of second kind and order zero and one, respectively. A corresponding solution

76 in the time domain $s(r_D, z_D, p_D)$, is obtained through numerical inversion of the Laplace

transform by means of an algorithm due to Crump [1976] as modified by de Hoog et. al. [1982].

Whereas standard inversion with respect to p is done over a time interval [0,t], we do the

inversion with respect to p_D over a unit dimensionless time (corresponding to p_D^{-1}) interval

[0,1], regardless of what t_c is.

Vertically Averaged Drawdown

Drawdown in a piezometer or observation well that penetrates the aquifer between dimensionless depths $z_{D1} = z_1/b$ and $z_{D2} = z_2/b$ at a dimensionless radial distance r_D from the pumping well (Figure 1) is obtained by averaging the point drawdown over this interval according to

86
$$\overline{s}_{z_{D2}-z_{D1}}(r_D, p_D) = \frac{1}{z_{D2}-z_{D1}} \int_{z_{D1}}^{z_{D2}} s(r_D, z_D, p_D) dz_D.$$

87 (7)

72

79

80

81

$$\overline{s}_{z_{D2}-z_{D1}}(r_{D}, p_{D}) = \frac{Qt}{4\pi K_{r}b} \left\{ \frac{2}{p_{D}} \frac{K_{0}(\phi_{0})}{r_{wD}\phi_{0}K_{1}(r_{wD}\phi_{0}) + \frac{C_{wD}}{2(l_{D}-d_{D})}r_{wD}^{2}\phi_{0}^{2}K_{0}(r_{wD}\phi_{0})} + \frac{4}{p_{D}\pi^{2}(l_{D}-d_{D})(z_{D2}-z_{D1})} \sum_{n=1}^{\infty} \frac{K_{0}(\phi_{n})\left[\sin(n\pi l_{D}) - \sin(n\pi d_{D})\right]\left[\sin(n\pi z_{D2}) - \sin(n\pi z_{D1})\right]/n^{2}}{r_{wD}\phi_{n}K_{1}(r_{wD}\phi_{n}) + \frac{C_{wD}}{2(l_{D}-d_{D})}r_{wD}^{2}\phi_{0}^{2}K_{0}(r_{wD}\phi_{n})} \right\}$$
(8)

90 Reduction to Solution of *Papadopoulos and Cooper* [1967]

- When the pumping well is fully penetrating $l_D = 1$, $d_D = 0$ and (6) reduces to the
- ocrresponding Laplace domain solution of *Papadopoulos and Cooper* [1967],

93
$$\overline{s}(r_D, p_D) = \frac{Qt}{4\pi K_r b} \left\{ \frac{2}{p_D} \frac{K_0\left(\sqrt{\frac{p_D}{t_s}}\right)}{r_{wD}\sqrt{\frac{p_D}{t_s}}K_1\left(r_{wD}\sqrt{\frac{p_D}{t_s}}\right) + \frac{C_{wD}}{2}r_{wD}^2 \frac{p_D}{t_s}K_0\left(r_{wD}\sqrt{\frac{p_D}{t_s}}\right)} \right\}$$
 (9)

94 Reduction to Solution's of Yang et.al. [2006], Hantush [1964] and Theis [1935]

- When the pumping well has finite diameter $(r_w \neq 0)$ but negligible or no wellbore
- storage $(C_{wD} \rightarrow 0)$, (6) reduces to the solution of *Yang et. al.* [2006] in Laplace space,

$$\overline{s}(r_{D}, z_{D}, p_{D}) = \frac{Qt}{4\pi K_{r}b} \left\{ \frac{2}{p_{D}} \frac{K_{0}(\phi_{0})}{r_{wD}\phi_{0}K_{1}(r_{wD}\phi_{0})} + \frac{4}{p_{D}\pi(l_{D} - d_{D})} \sum_{n=1}^{\infty} \frac{K_{0}(\phi_{n}) \left[\sin(n\pi l_{D}) - \sin(n\pi d_{D}) \right] \cos(n\pi z_{D}) / n}{r_{wD}\phi_{n}K_{1}(r_{wD}\phi_{n})} \right\}$$
(10)

- When the pumping well has small diameter $(r_w \rightarrow 0)$, (6) reduces to *Hantush*'s [1964]
- solution in Laplace space due to the fact that $xK_1(x) \rightarrow 1$ and $x^2K_0(x) \rightarrow 0$ as $x \rightarrow 0$,

$$100 \qquad \overline{s}\left(r_{D}, z_{D}, p_{D}\right) = \frac{Qt}{4\pi K_{r}b} \left\{ \frac{2}{p_{D}} K_{0} \left(\sqrt{\frac{p_{D}}{t_{s}}}\right) + \frac{4}{p_{D}\pi} \sum_{n=1}^{\infty} \frac{K_{0}\left(\phi\right) \left[\sin\left(n\pi l_{D}\right) - \sin\left(n\pi d_{D}\right)\right] \cos\left(n\pi z_{D}\right)}{n\left(l_{D} - d_{D}\right)} \right\}$$

$$101 \qquad . \tag{11}$$

It is well established and easily verified that the latter in turn reduces to the *Theis* [1935] solution in Laplace space when the pumping well becomes fully penetrating $(d_D = 0, l_D = 1)$,

$$104 \qquad \overline{s}\left(r_{D}, p_{D}\right) = \frac{Qt}{4\pi K_{r}b} \left\{ \frac{2}{p_{D}} K_{0} \left(\sqrt{\frac{p_{D}}{t_{s}}}\right) \right\} \tag{12}$$

3. RESULTS AND DISCUSSION

To investigate the effect of partial penetration, wellbore storage and anisotropy on drawdown we consider a pumping well of dimensionless radius $r_w/b = 0.02$.

Drawdown in pumping well

We start by considering drawdown in a pumping well penetrating the upper half $(d_D = 0.0, l_D = 0.5)$ of an isotropic aquifer with $K_D = 1.0$. Figure 2 compares the variation of dimensionless drawdown $s_D(r_D, z_D, t_s) = (4\pi K_r b/Q)s(r_D, z_D, t_s)$ in the pumping well with dimensionless time t_s using different analytical solutions when $C_{wD} = 1.0 \times 10^2$. At early time water is derived entirely from wellbore storage, rendering dimensionless drawdown linearly proportional to dimensionless time (forming a line with unit slope on log-log scale); our solution and that of *Papadopulos and Cooper [1967]* reflect this clearly. Solutions that do not account for wellbore storage predict a much earlier rise in drawdown. Whereas the *Papadopulos and Cooper [1967]* solution approaches that of *Theis* [1935] at later dimensionless time, ours approaches that of *Hantush* [1964] as the effects of finite radius and wellbore storage dissipate. The solution of *Yang et al.* [2006], which considers only the first effect, exhibits an earlier rise

in dimensionless drawdown than do any of the other solutions, eventually coinciding with that of *Hantush* [1964]. Dimensionless drawdown in the pumping well at late dimensionless time exceeds that predicted by solutions which ignore partial penetration.

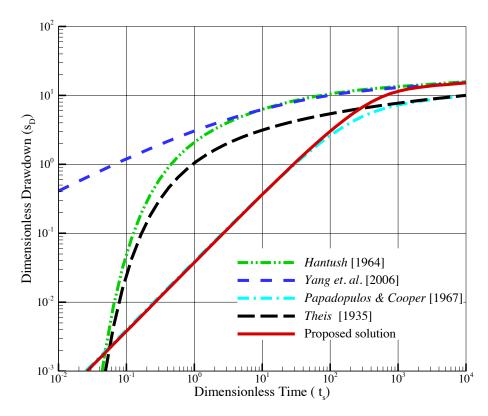


Figure 2: Dimensionless drawdown in pumping well versus dimensionless time, computed by various analytical solutions when $C_{wD} = 1.0 \times 10^2$, $d_D = 0.0$, $l_D = 0.5$ and $K_D = 1.0$.

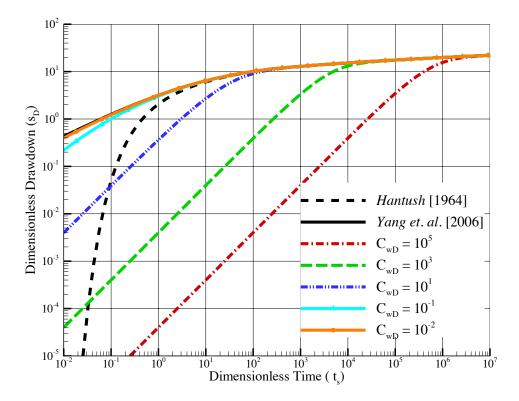


Figure 3: Dimensionless drawdown in pumping well versus dimensionless time for various values of dimensionless wellbore storage C_{wD} when $d_D = 0.0$, $l_D = 0.5$ and $K_D = 1.0$.

Figure 3 shows how dimensionless drawdown in the pumping well varies with dimensionless time t_s for different values of the dimensionless wellbore storage coefficient, C_{wD} . As with the solution of *Papadopulos and Cooper* [1967], the larger is C_{wD} the longer does wellbore storage impact drawdown in the pumping well. As C_{wD} diminishes our solution approaches that of *Yang et.al* [2006].

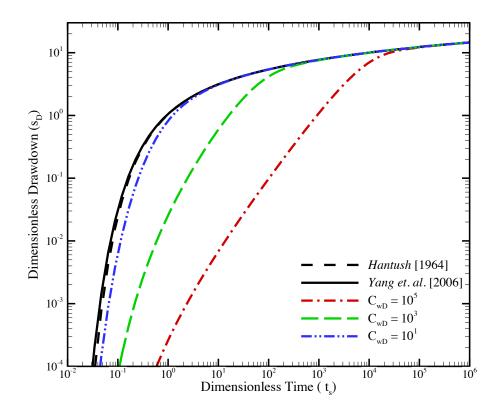


Figure 4: Dimensionless drawdown at $z_D = 0.5$ and $r_D = 0.2$ versus dimensionless time for various values of dimensionless wellbore storage C_{wD} when $l_D = 0.5$ and $K_D = 1.0$.

Drawdown in piezometer

Figure 4 shows dimensionless time-drawdown variations at dimensionless radial distance $r_D = 0.2$ from the axis of the pumping well and dimensionless elevation $z_D = 0.5$ (midway between the horizontal no-flow boundaries) for different values of C_{wD} under the above conditions. When C_{wD} is large, the early dimensionless time-drawdown curve on log-log scale is nearly linear with a unit slope, reflecting a strong effect of storage in the pumping well on early drawdown in a nearby piezometer. As C_{wD} diminishes this effect becomes less discernible, the curve becoming nonlinear and steeper. The curve tends asymptotically toward the solution of $Yang\ et\ al.\ [2006]$, which in turn is very close to that of $Hantush\ [1964]$ due to the small dimensionless radius we have assigned to the pumping well in our example.

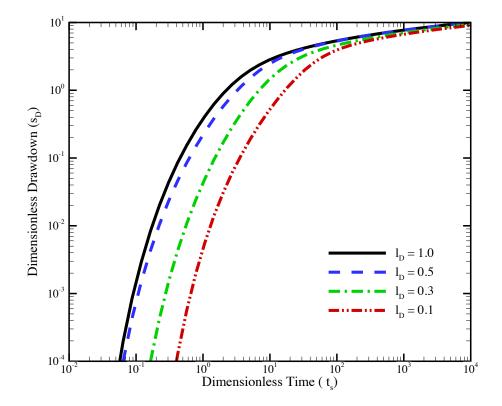


Figure 5: Dimensionless drawdown at $z_D = 0.5$ and $r_D = 0.2$ versus dimensionless time for various screen lengths l_D when $C_{wD} = 1.0 \times 10^2$, $d_D = 0$ and $K_D = 1.0$.

Figure 5, corresponding to the case where $C_{wD} = 1.0 \times 10^2$, shows that dimensionless drawdown at $r_D = 0.2$ and $z_D = 0.5$ increases when the pumping well is extended to the aquifer bottom ($d_D = 0.0, l_D = 1.0$ below the observation point) but decreases when this well becomes shallower; a similar trend is reflected in the solution of *Hantush* [1964]. Reducing the ratio K_D between vertical and horizontal hydraulic conductivity in the case of a well that is shallower than the observation point ($d_D = 0.0, l_D = 0.25$) likewise causes dimensionless drawdown at this point to diminish (Figure 6).

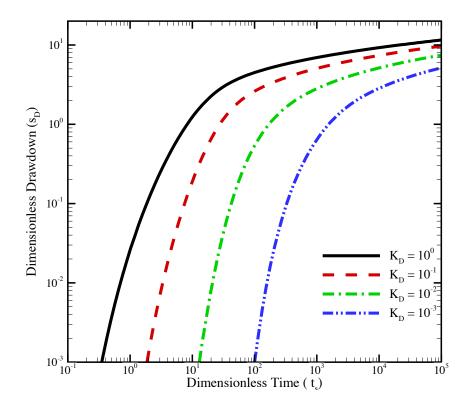


Figure 6: Dimensionless drawdown at $z_D = 0.5$ and $r_D = 0.2$ versus dimensionless time for various anisotropy ratios $K_D = K_z / K_r$ when $C_{wD} = 1.0 \times 10^2$, $d_D = 0.0$ and $l_D = 0.25$.

Figure 7 illustrates the impact of dimensionless radial distance from the pumping well on dimensionless time-drawdown at $z_D = 0.5$ when $d_D = 0.0$, $l_D = 0.25$, $K_D = 1$ and $C_{wD} = 1.0 \times 10^2$. As this distance increases the effects of both wellbore storage and partial penetration diminish, the dimensionless time-drawdown response in the aquifer approaching that predicted by *Theis* [1935].

4. SUMMARY AND CONCLUSION

A new analytical solution has been developed for a partially penetrating well of finite diameter with storage pumping at a constant rate from an anisotropic confined aquifer. Our solution unifies the solutions of *Papadopulos and Cooper* [1967], *Hantush* [1964], *Theis*

[1935] and *Yang et.al.* [2006] by accounting simultaneously for aquifer anisotropy, partial penetration and wellbore storage capacity of the pumping well under confined conditions. We used our solution to explore all three effects. Reducing the anisotropy ratio $K_D = K_z/K_r$ causes drawdown in the aquifer to decrease. Whereas the effect of partial penetration decreases with increasing distance from the pumping well, that of wellbore storage diminishes with distance and time.

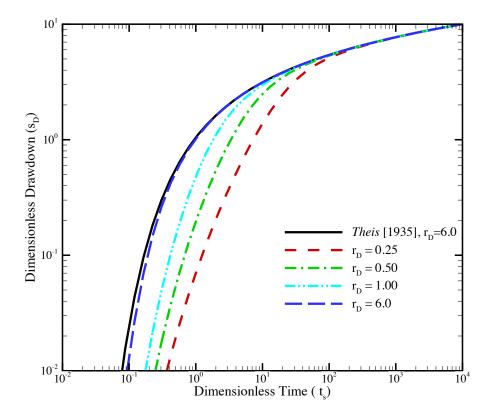


Figure 7: Dimensionless drawdown versus dimensionless time at $z_D = 0.5$ and various values of $r_D = r/b$ when $C_{wD} = 1.0 \times 10^3$, $d_D = 0.0$, $l_D = 0.25$ and $K_D = 1.0$.

Appendix A: Laplace transformed drawdown

Introducing a new variable $r' = r(K_z/K_r)^{1/2} = rK_D^{-1/2}$ and taking Laplace transform of (1)

180 – (5) gives

181
$$\frac{\partial^2 \overline{s}}{\partial r'^2} + \frac{1}{r'} \frac{\partial \overline{s}}{\partial r'} + \frac{\partial^2 \overline{s}}{\partial z^2} = \frac{S_s}{K_z} p \overline{s}$$
 (A1)

subject to

184
$$\overline{s}(\infty, z, p) = 0$$
 (A2)

185

186
$$\frac{\partial \overline{s}}{\partial z} = 0 \quad \text{at } z = 0 \& z = b$$
 $r > r_w$ (A3)

$$2\pi \left(l - d\right) K_r r_w' \left(\frac{\partial \overline{s}}{\partial r'}\right)_{r' = r_w'} - C_w p \overline{s}_{r' = r_w'} = -\frac{Q}{p} \qquad d < z < l$$
(A4)

188
$$r'_{w} \left(\frac{\partial \overline{s}}{\partial r'}\right)_{r'=r'_{w}} = 0 \qquad 0 < z < d & l < z < b$$
 (A5)

189

Defining the finite cosine transform of $\overline{s}(r', z, p)$ as (*Churchill*, 1958, p.354-355)

191
$$f_c\left\{\overline{s}\left(r',z,p\right)\right\} = \overline{s}_c\left(r',n,p\right) = \int_0^b \overline{s}\left(r',z,p\right)\cos\left(n\pi z/b\right) dz \qquad n = 0,1,2,...$$
 (A6)

192 with inverse

193
$$\overline{s}(r',z,p) = \frac{1}{b}\overline{s}_c(r',0,p) + \frac{2}{b}\sum_{n=1}^{\infty}\overline{s}_c(r',n,p)\cos(n\pi z/b)$$
 (A7)

implies that, by virtue of (A3),

$$195 f_c \left\{ \frac{\partial^2 \overline{s}}{\partial z^2} \right\} = -\left(\frac{n\pi}{b} \right)^2 \overline{s}_c (r', n, p) + \left(-1 \right)^n \frac{\partial \overline{s} (r', z, p)}{\partial z} \bigg|_{z=b} - \frac{\partial \overline{s} (r', z, p)}{\partial z} \bigg|_{z=0} = -\left(\frac{n\pi}{b} \right)^2 \overline{s}_c (r', n, p)$$

$$(A7)$$

Hence finite cosine transformation of (A1) - (A5) leads to

198
$$\frac{\partial^2 \overline{s}_c}{\partial r'^2} + \frac{1}{r'} \frac{\partial \overline{s}_c}{\partial r'} - \left[\frac{p}{K_z / S_s} + \left(\frac{n\pi}{b} \right)^2 \right] \overline{s}_c = 0$$
 (A8)

$$\overline{S}_{c}(\infty, n, p) = 0 \tag{A9}$$

$$2\pi (l-d) K_r r'_w \left(\frac{\partial \overline{s}_c}{\partial r'}\right)_{r'=r'_w} - p C_w (\overline{s}_c)_{r'=r'_w} = -\frac{Q}{p} \int_d^l \cos(n\pi z/b) dz$$

$$= -\frac{Q}{p} (b/n\pi) \left[\sin(n\pi l/b) - \sin(n\pi d/b) \right]$$

201 (A10)

The general solution of (A8) is

$$\overline{s}_{c}(r',n,p) = AK_{0}(Nr') + BI_{0}(Nr')$$
(A11)

where $N^2 = \frac{p}{K_z/S_s} + (n\pi/b)^2$, I_0 and K_0 being modified Bessel functions of first and second

kind, respectively, and of zero order. By virtue of (A9) B = 0. Substituting this and (A11) into

206 (A10), noting that $\partial K_0(Nr')/\partial r' = -NK_1(Nr')$, solving for A and substituting back into (A11) yields

207
$$\overline{s}_{c}(r', n, p) = \frac{Q}{p} \frac{(b/n\pi) \left[\sin(n\pi l/b) - \sin(n\pi d/b) \right]}{2\pi (l-d) K_{r} N r'_{w} K_{1}(N r'_{w}) + p C_{w} K_{0}(N r'_{w})} K_{0}(N r').$$
(A12)

Noting that
$$\lim_{n\to 0} \left[l \frac{\sin(n\pi l/b)}{n\pi l/b} - d \frac{\sin(n\pi d/b)}{n\pi d/b} \right] = l - d$$
 one gets

209
$$\overline{S}_{c}(r',0,p) = \frac{Q}{p} \frac{K_{0}\left(r'\sqrt{\frac{p}{K_{z}/S_{s}}}\right)}{2\pi K_{r}r'_{w}\sqrt{\frac{p}{K_{z}/S_{s}}}K_{1}\left(r'_{w}\sqrt{\frac{p}{K_{z}/S_{s}}}\right) + \frac{pC_{w}}{(l-d)}K_{0}\left(r'_{w}\sqrt{\frac{p}{K_{z}/S_{s}}}\right)}.$$
(A13)

210 This allows obtaining the inverse Fourier cosine transform of (A12),

$$\overline{s}(r',z,p) = \frac{1}{b} \frac{Q}{K_{r}p} \frac{K_{0}\left(r'\sqrt{\frac{p}{K_{z}/S_{s}}}\right)}{2\pi r'_{w}\sqrt{\frac{p}{K_{z}/S_{s}}}K_{1}\left(r'_{w}\sqrt{\frac{p}{K_{z}/S_{s}}}\right) + \frac{pC_{w}}{K_{r}(l-d)}K_{0}\left(r'_{w}\sqrt{\frac{p}{K_{z}/S_{s}}}\right)} + \frac{2}{b} \frac{Q}{K_{r}p} \sum_{n=1}^{\infty} \frac{(b/n\pi)\left[\sin\left(n\pi l/b\right) - \sin\left(n\pi d/b\right)\right]}{2\pi(l-d)Nr'_{w}K_{1}\left(Nr'_{w}\right) + \frac{pC_{w}}{K_{r}}K_{0}\left(Nr'_{w}\right)} \cos\left(n\pi z/b\right)K_{0}\left(Nr'\right)} \tag{A14}$$

Recalling that $r' = rK_D^{1/2}$ and rewriting (A14) in dimensionless form yields (6).

213 REFERENCES

- 214 Churchil, R.V. (1958), *Operational Mathematics*, 2nd edition, McGraw Hill, New York.
- 215 Crump, K. S. (1976), Numerical inversion of Laplace transforms using a Fourier series
- approximation, J. Assoc. Comput. Mach., vol.23, issue 1, 89–96.
- de Hoog, F. R., J. H. Knight, and A. N. Stokes (1982), An improved method for numerical
- inversion of Laplace transforms, SIAM J. Sci. Stat. Comput., vol. 3 Issue 3, 357–366,
- doi:10.1137/0903022.
- Dougherty, D. E., and D. K. Babu (1984), Flow to a partially penetrating well in a double-
- porosity reservoir, Water Resour. Res., 20, 1116-1122.
- Hantush, M. S. (1964), Hydraulics of wells, *Adv. Hydrosci.*, *1*, 281–442.
- Kucuk, F., and W. E. Brigham (1979), Transient flow in elliptical systems, Soc. Pet, Eng. J., 19,
- 401-410, doi:10.2118/7488-PA.
- Mathias, S. A., and A. P. Butler (2007), Flow to a finite diameter well in a horizontally

- anisotropic aguifer with wellbore storage, *Water Resour. Res.*, 43, W07501, doi:
- 227 10.1029/2006WR005839.
- Moench, A. F. (1997), Flow to a well of finite diameter in a homogenous anisotropic water table
- aquifer, Water Resour. Res., 33, No.6, 1397- 1407.
- Moench, A. F. (1998), Correction to "Flow to a well of finite diameter in a homogenous
- anisotropic water table aquifer", *Water Resour. Res.*, 34, No.9, 2431-2432.
- Papadopulos, I. S., and H. H. Cooper Jr. (1967), Drawdown in a well of large diameter, *Water*
- 233 Resour. Res., 3(1), 241–244.
- Theis, C. V. (1935), The relationship between the lowering of the piezometric surface and rate
- and duration of discharge of a well using groundwater storage, Eos Trans. AGU, 16,
- 236 519–524.
- Van Everdingen, A.F. and Hurst, W. (1949), The application of the Laplace transformation to
- flow problems in reservoirs, *Trans AIME*, 186, 305-324.
- Yang, S.-Y., H.-D. Yeh, and P.-Y. Chiu (2006), A closed form solution for constant flux
- pumping in a well under partial penetration condition, *Water Resour. Res.*, 42, W05502,
- doi:10.1029/2004WR003889.